REMARKS ON POINCARÉ DUALITY OF MATROIDS

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ABSTRACT. Adiprasito, Huh, and Katz show Poincaré duality of matroids using matroidal flips, which are purely combinatorial notions. Later, Hsiao, Karu, and Yang investigate the Hodge theory of matroids via star subdivisions, which correspond to blow-ups of toric varieties. This is an expository paper reviewing a simple proof of Poincaré duality of matroids using their method of star subdivisions and Keel's lemma from algebraic geometry.

1. INTRODUCTION

Let n, r be non-negative integers. In this paper, a *matroid* M refers to a loopless matroid of rank $r+1$ on an $(n+1)$-set E. Write $\mathcal{L}(M)$ for the lattice of flats of M. Let $\bigoplus_{i\in E} \mathbf{Z}e_i$ be the free abelian group with E as its basis. Put $e_S \coloneqq \sum_{i\in S} e_i$ for a subset $S \subset E$, $N_E \coloneqq \left(\bigoplus_{i\in E} \mathbf{Z}e_i\right)/\mathbf{Z}e_E$, and $N_{E,\mathbf{R}} \coloneqq N_E \otimes_{\mathbf{Z}} \mathbf{R}$. The *Bergman fan* Σ_M of a matroid M is the pure r-dimensional unimodular fan

$$\left\{ \operatorname{Cone}(e_F \,:\, F \in \mathcal{F}) \,:\, \mathcal{F} \subset \mathcal{P}(M) \text{ is a flag} \right\}$$

in $N_{E,\mathbf{R}}$. Here, $\mathcal{P}(M) \coloneqq \mathcal{L}(M) \setminus \{\varnothing, E\}$, and $\mathcal{F} \subset \mathcal{P}(M)$ is called a *flag* if it is linearly ordered with respect to the inclusion relation. Associated to Σ_M, we can consider the n-dimensional smooth quasi-projective complex toric variety X_{Σ_M}, which is possibly non-compact. Then, the *Chow ring* $A^\bullet(M)$ of M is defined to be the Chow ring $A^\bullet(X_{\Sigma_M})$ of X_{Σ_M}. Adiprasito, Huh, and Katz show Poincaré duality of M in the following sense.

Theorem 1 ([1, Theorem 6.19]). *For a matroid M of rank $r+1$, $A^\bullet(M)$ satisfies Poincaré duality of dimension r, i.e., it satisfies*

1) *there exists a group isomorphism* $\deg \colon A^r(M) \xrightarrow{\sim} \mathbf{Z}$*, called a* degree map*,*
2) *for every integer $q > r$, we have $A^q(M) = 0$, and*
3) *for every integer $q \leqq r$, the map*

$$A^{r-q}(M) \to \operatorname{Hom}(A^q(M), \mathbf{Z}); \quad x \mapsto \big(y \mapsto \deg(xy)\big)$$

is a group isomorphism, i.e., the multiplication $A^q(M) \times A^{r-q}(M) \to A^r(M) \xrightarrow{\sim} \mathbf{Z}$ is a perfect pairing.

The key concepts of their proof are matroidal flips, which are purely combinatorial technical notions. On the other hand, motivated by [3], Hsiao, Karu, and Yang [5] calculate mixed volumes of matroids using the deletion of matroids. For their computation, they utilize the following important fact. For a non-coloop $a \in E$ of M and the deletion $M\backslash a$, the projection $\pi \colon N_E \to N_{E\backslash\{a\}}$ naturally induces a morphism of fans $\pi \colon \Sigma_M \to \Sigma_{M\backslash a}$, and this morphism factors as

$$(\clubsuit) \qquad \Sigma_M = \Sigma_m \xrightarrow{\pi_m} \Sigma_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} \Sigma_1 \xrightarrow{\pi_1} \Sigma_0 \xrightarrow{\pi_0} \Sigma_{M\backslash a},$$

where the morphism $\pi_0 \colon \Sigma_0 \to \Sigma_{M\backslash a}$ is induced by the projection $\pi \colon N_E \to N_{E\backslash\{a\}}$, $\pi_i \colon \Sigma_i \to \Sigma_{i-1}$ is induced by the identity map, and Σ_i is a star subdivision of Σ_{i-1} at a 2-dimensional unimodular cone for each $i = 1, \dots, m$. Using the factorization ($\clubsuit$), Hsiao [4] gives a simpler combinatorial proof that $A^\bullet(M) \otimes_{\mathbf{Z}} \mathbf{R}$ satisfies the hard Lefschetz property, the Hodge-Riemann relations, and therefore Poincaré duality. We note that Amini and Piquerez [2, Section 1.9] also sketched a similar proof of these properties for matroids via star subdivisions, relying on the weak factorization theorem for tropical fans. In §3, we will give a proof of Theorem 1 in the language of algebraic geometry using ($\clubsuit$) and Keel's lemma.

2. Keel's lemma

In this section, we recall an algebro-geometric lemma, which is known as Keel's lemma.

Theorem 2 (Keel's lemma, cf. [6, Appendix, Theorem 1]). *Let X be a smooth variety, $i\colon Z \hookrightarrow X$ a smooth closed subvariety of codimension z, $\pi\colon \widetilde{X} \to X$ the blow-up of X along Z, and E the exceptional divisor, that is, we consider the pullback diagram*

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"'] & X. \end{tikzcd}$$

If the pullback map $i^*\colon A^\bullet(X) \to A^\bullet(Z)$ is surjective, then the map $\pi^*\colon A^\bullet(X)[T] \to A^\bullet(\widetilde{X});\ T \mapsto -[E]$ induces an isomorphism

$$\frac{A^\bullet(X)[T]}{(P(T), T\cdot \operatorname{Ker} i^*)} \xrightarrow{\ \sim\ } A^\bullet(\widetilde{X})$$

where $P(T) \in A^\bullet(X)[T]$ is a monic polynomial of degree z. In particular, we have a decomposition

$$(\diamond) \qquad A^q(\widetilde{X}) = A^q(X) \oplus [E]\cdot A^{q-1}(Z) \oplus \cdots \oplus [E]^{z-1}\cdot A^{q-z+1}(Z)$$

for all q, where we embed $\pi^*\colon A^\bullet(X) \hookrightarrow A^\bullet(\widetilde{X})$, and for all positive integers p, q,

$$A^{q-p}(Z) \xrightarrow[(i^*)^{-1}]{\sim} A^{q-p}(X)/\operatorname{Ker} i^* \xhookrightarrow[{[E]^p\cdot(-)}]{} A^q(\widetilde{X}).$$

Now we go back to our toric case. Let Σ be a unimodular fan in $N_{E,\mathbf{R}}$. For a z-dimensional cone $\sigma \in \Sigma$, let Σ_σ denote the star fan of Σ at σ, and let $\Sigma^*(\sigma)$ denote the star subdivision of Σ at σ. The orbit closure $V(\sigma)$ associated to σ is the smooth toric subvariety X_{Σ_σ} of codimension z, and $\pi\colon X_{\Sigma^*(\sigma)} \to X_\Sigma$ is the blow-up of X_Σ along $V(\sigma)$. Since the ring $A^\bullet(X_\Sigma)$ is generated by the classes of torus-invariant divisors, the pullback homomorphism $A^\bullet(X_\Sigma) \to A^\bullet(X_{\Sigma_\sigma})$ is always surjective, so the assumption of Theorem 2 is satisfied if we substitute $X = X_\Sigma$, $Z = V(\sigma) = X_{\Sigma_\sigma}$, $\widetilde{X} = X_{\Sigma^*(\sigma)}$. We also note that the q-th degree part $A^q(X_\Sigma)$ is a finitely generated abelian group.

Corollary 3. *We further assume that $A^\bullet(X_\Sigma)$ and $A^\bullet(X_{\Sigma_\sigma})$ satisfy Poincaré duality of dimension r and $r-z$ respectively. Then $A^\bullet(X_{\Sigma^*(\sigma)})$ satisfies Poincaré duality of dimension r.*

Proof. The decomposition $(\diamond)$ tells us that $A^r(X_{\Sigma^*(\sigma)}) \cong \mathbf{Z}$ and $A^q(X_{\Sigma^*(\sigma)}) = 0$ for each $q > r$. By Poincaré duality of $A^\bullet(X)$ and $A^\bullet(Z)$, each summand of $(\diamond)$ is a finitely generated free abelian group. With an apropriately ordered basis chosen from each summand, the representation matrix of the bilinear pairing $A^q(X_{\Sigma^*(\sigma)}) \times A^{r-q}(X_{\Sigma^*(\sigma)}) \to \mathbf{Z}$ is a block lower triangular matrix

$$\begin{bmatrix} P_0 & O & \cdots & \cdots & O \\ O & P_1 & \ddots & \ddots & \vdots \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ O & * & * & * & P_{z-1} \end{bmatrix}.$$

Poincaré duality of $A^\bullet(X_\Sigma)$ and $A^\bullet(X_{\Sigma_\sigma})$ imply that $\det P_q = \pm 1$ for all $q = 0, 1, \ldots, z-1$, and therefore the above representation matrix has determinant ± 1. ∎

3. A proof of Poincaré duality by star subdivisions

In this section, we prove Theorem 1 by induction on n. Before starting the proof, we recall the definition of a coloop.

Definition 4. An element $a \in E$ is called a *coloop* of M if the subset $E \setminus \{a\}$ is a flat of M, or equivalently, $\dim \Sigma_{M\backslash a} < \dim \Sigma_M$.

If all elements of E are coloops of M, then M must be the Boolean matroid on E. The toric variety X_{Σ_M} is a smooth projective complex variety of dimension $r = n$. By the Jurkiewicz-Danilov theorem, the cycle map $A^\bullet(X_{\Sigma_M}) \to H^{2\bullet}(X_{\Sigma_M}, \mathbf{Z})$ is an isomorphism, and thus Poincaré duality of $A^\bullet(M)$ follows in the usual sense. In particular, this establishes the statement for $n = 0$.

We assume that $n > 0$ and there exists a non-coloop $a \in E$ of M. If the singleton $\{a\}$ is not a flat of M, then the deletion map $d : \mathcal{L}(M) \to \mathcal{L}(M\backslash a)$; $F \mapsto F \setminus \{a\}$ is bijective, and $\Sigma_M \subset (e_a - e_b)^\perp$ for some element b parallel to a. This implies that $X_{\Sigma_M} \cong X_{\Sigma_{M\backslash a}} \times \mathbf{G}_\mathrm{m}$, and hence $A^\bullet(M) \cong A^\bullet(M\backslash a)$, where the right-hand side satisfies Poincaré duality of dimension r by the inductive hypothesis. Hence we may assume that $\{a\}$ is a flat of M. Now we construct the factorization (♣). Logically, the set $\mathcal{P}(M)$ can be partitioned into four disjoint subsets

$$\begin{aligned}\mathcal{P}(M) = &\big\{ F : a \notin F,\ F \cup \{a\} \in \mathcal{P}(M) \big\} \sqcup \big\{ F : a \notin F,\ F \cup \{a\} \notin \mathcal{P}(M) \big\} \\ &\sqcup \big\{ F : a \in F,\ F \setminus \{a\} \in \mathcal{P}(M) \big\} \sqcup \big\{ F : a \in F,\ F \setminus \{a\} \notin \mathcal{P}(M) \big\} \\ =: &\ \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \mathcal{P}_3 \sqcup \mathcal{P}_4.\end{aligned}$$

Under the deletion map $d : \mathcal{P}(M) \to \mathcal{P}(M\backslash a)$, we have

$$\mathcal{P}(M\backslash a) = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup d(\mathcal{P}_4), \qquad d(\mathcal{P}_3) \subset \mathcal{P}_1, \qquad \mathcal{P}(M/a) = d(\mathcal{P}_3) \sqcup d(\mathcal{P}_4).$$

Consider the inverse map $d^{-1} : \mathcal{P}(M\backslash a) \to \mathcal{P}(M) \setminus \mathcal{P}_3$. We construct a fan Σ_0 in $N_{E,\mathbf{R}}$ by

$$\begin{aligned}\Sigma_0 := &\big\{ \operatorname{Cone}(e_{d^{-1}(F)} : F \in \mathcal{F}) : \mathcal{F} \underset{\text{flag}}{\subset} \mathcal{P}(M\backslash a) \big\} \\ &\cup \big\{ \operatorname{Cone}(e_{d^{-1}(F)} : F \in \mathcal{F}) + \operatorname{Cone}(e_a) : \mathcal{F} \underset{\text{flag}}{\subset} \mathcal{P}(M/a) \big\}\end{aligned}$$

(the latter cones are intended to recover the rays associated to $\mathcal{P}_3$). Observing that Σ_0 has exactly one more ray than $\Sigma_{M\backslash a}$ (the ray is $\operatorname{Cone}(e_a)$), one can show that the projection $\pi_0 : \Sigma_0 \to \Sigma_{M\backslash a}$ induces an isomorphism $\pi_0^* : A^\bullet(M\backslash a) \xrightarrow{\sim} A^\bullet(X_{\Sigma_0})$. We order the flats $d(\mathcal{P}_3) = \{F_1, F_2, \dots, F_m\}$ so that $F_i \subset F_j$ implies that $i \leqq j$. Recall that, since $d(\mathcal{P}_3) \subset \mathcal{P}_1$, we have $d^{-1}(F_i) = F_i$ for all i, and hence $\operatorname{Cone}(e_a, e_{F_i}) \in \Sigma_0$ for all i. Let Σ_1 be the star subdivision of Σ_0 at $\operatorname{Cone}(e_{F_m}, e_a)$. Σ_1 has an extra ray $\operatorname{Cone}(e_{F_m \cup \{a\}})$, and all new cones arising from this subdivision are of the form

$$\begin{aligned}\operatorname{Cone}(e_{d^{-1}(F)} : F \in \mathcal{F}) + \operatorname{Cone}(e_{F_m \cup \{a\}}) \quad &\text{where } \mathcal{F} \text{ is a flag of } \mathcal{P}(M/a), \text{ or} \\ \operatorname{Cone}(e_{d^{-1}(F)} : F \in \mathcal{F}) + \operatorname{Cone}(e_a) + \operatorname{Cone}(e_{F_m \cup \{a\}}) \quad &\text{where } \mathcal{F} \text{ is a flag of } \mathcal{P}(M/a) \setminus \{F_m\}.\end{aligned}$$

The cones of the former form are already cones of Σ_M. By recursively constructing the star subdivision Σ_{i+1} of Σ_i at $\operatorname{Cone}(e_{F_{m-i}}, e_a)$ for each $i = 1, \dots, m-1$, we obtain $\Sigma_m = \Sigma_M$ and the desired factorization

$$(\clubsuit) \qquad \Sigma_M = \Sigma_m \xrightarrow{\pi_m} \Sigma_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} \Sigma_1 \xrightarrow{\pi_1} \Sigma_0 \xrightarrow{\pi_0} \Sigma_{M\backslash a}.$$

Finally, the star fan of Σ_i at $\operatorname{Cone}(e_{F_{m-i}}, e_a)$ is isomorphic to the product $\Sigma_{[\varnothing, F_{m-i}]} \times \Sigma_{[F_{m-i} \cup \{a\}, E]}$, where $[\varnothing, F_{m-i}]$ denotes the restriction of M to F_{m-i} and $[F_{m-i} \cup \{a\}, E]$ denotes the contraction of M by $F_{m-i} \cup \{a\}$. This implies that its Chow ring is isomorphic to $A^\bullet([\varnothing, F_{m-i}]) \otimes A^\bullet([F_{m-i} \cup \{a\}, E])$, which satisfies Poincaré duality of dimension $(\operatorname{rk}_M(F_{m-i}) - 1) + (\operatorname{crk}_M(F_{m-i} \cup \{a\}) - 1) = r - 2$ by the inductive hypothesis. Applying Corollary 3 with $z = 2$, we conclude that $A^\bullet(M)$ satisfies Poincaré duality of dimension r. ∎

We give some remarks of the proof.

Remark 5. If we subdivide Σ_0 at $\mathrm{Cone}(e_{F_1}, e_a)$, the resultant fan differs from Σ_M. For instance, when $F_1 \subset F_2$, $\mathrm{Cone}(e_{F_1}, e_{F_2}, e_a)$ is divided into $\mathrm{Cone}(e_{F_1 \cup \{a\}}, e_{F_2}, e_a)$ and $\mathrm{Cone}(e_{F_1 \cup \{a\}}, e_{F_1}, e_{F_2})$. The former has a face $\mathrm{Cone}(e_{F_1 \cup \{a\}}, e_{F_2})$, which remains in the final fan and is not a cone of Bergman fan.

Remark 6. In the language of tropical geometry, the fan Σ_0 is known as the tropical modification of the tropical fan $\Sigma_{M \backslash a}$ along the tropical divisor $\Sigma_{M/a}$ with respect to a suitable piecewise linear function on $\Sigma_{M \backslash a}$. Then the isomorphism $\pi^* : A^\bullet(M \backslash a) \to A^\bullet(X_{\Sigma_0})$ follows from [2, Theorem 6.4].

Remark 7. Amini and Piquerez [2, Theorem 7.9] show that X_{Σ_M} and X_{Σ_0} can be connected by a finite sequence of toric blowups and blowdowns (weak factorization for tropical fans). Therefore, the sequence $\Sigma_m \to \Sigma_{m-1} \to \cdots \to \Sigma_0$ in (♣) provides a concrete example of this theorem for matroids. Furthermore, it shows that X_{Σ_M} can be obtained from X_{Σ_0} by a finite sequence of blowups.

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